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HOMOLOGY AND COHOMOLOGY OF Π-ALGEBRAS

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ABSTRACT. We study a type of homological algebra associated to the collection of *all* homotopy groups of a space (just as the theory of group homology is associated to the fundamental group).

1. INTRODUCTION

1.1. **Summary.** In [3] we started an investigation of **Π**-algebras (i.e., (≥ 1)-graded groups, together with an action of the primary homotopy operations) by constructing an *enveloping ring* of a **Π**-algebra, which generalized the integral group ring of a group as well as the enveloping algebra of a connected graded rational Lie algebra. In the present paper we

(i) use the enveloping ring construction to define the *homology and cohomology of* **Π**-*algebras* in a manner which generalizes the usual homology and cohomology of groups and connected graded rational Lie algebras,
(ii) obtain *Serre spectral sequences* which, for a short exact sequence $* \to U \to W \to V \to *$ of **Π**-algebras, relate the homology and cohomology groups of U and V to those of W, and
(iii) discuss a slightly different, *Quillen-like approach* to the homology and cohomology of **Π**-algebras and note that (except in the bottom dimension) the resulting groups differ from the earlier ones (see (i)) by only a shift in dimension.

1.2. **A more detailed outline of the paper.** (i) *Homology of* **Π**-*algebras.* After fixing some notation and terminology (in §1.3) and a brief review of **Π**-algebras (in §2) and the enveloping ring functor E (in §3), we recall (in §4) some results of Quillen on simplicial modules over a simplicial ring R (with a slight change in notation: we write $\operatorname{Tor}_i^R(-,-)$ for Quillen's functors $\pi_i(-\otimes_R^L-)$). In §5 we then define the homology groups $H_i(X;M)$ of a **Π**-algebra X with coefficients in a right EX-module M, by

$$H_i(X;M) = \operatorname{Tor}^{EF.X}(M,Z), \qquad i \geq 0$$

(where $F.X$ denotes the standard free simplicial resolution (2.5) of X), and we obtain, for a short exact sequence $* \to U \to W \to V \to *$ of **Π**-algebras and a right EW-module M, a Serre spectral sequence with

$$E^2_{p,q} = H_p(V; H_q(U;M)) \Rightarrow H_{p+q}(W;M).$$

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(ii) *Cohomology of* **Π**-*algebras.* This requires "dualizing" the results of §§4 and 5. To do this we construct (in §6) functors $\operatorname{Ext}^i_R(-,-)$, which involve in the second variable rather curious "cosimplicial modules over the simplicial ring R". In §7 we then define the cohomology groups $H^i(X;N)$ of a **Π**-algebra X with coefficients in a left EX-module N, by

$$H^i(X,N) = \operatorname{Ext}^i_{EF.X}(Z,N), \qquad i \geq 0,$$

and we obtain, for a short exact sequence $* \to U \to W \to V \to *$ of **Π**-algebras and a left EW-module N, a Serre spectral sequence with

$$E_2^{p,q} = H^p(V; H^q(U;N)) \Rightarrow H^{p+q}(W;N).$$

(iii) *Quillen homology and cohomology.* The last two sections (§§8 and 9) are devoted to a Quillen-like approach to the homology and cohomology of a **Π**-algebra X. The key notion here is that of a strongly abelian group object over X, i.e., a diagram of **Π**-algebras

$$* \to B \xrightarrow{i} Y' \underset{j}{\overset{p}{\rightleftarrows}} X \to *$$

in which $pj = \mathrm{id}$, the right-pointing arrows form an exact sequence and B is strongly abelian (i.e., (2.2) just a (≥ 1)-graded abelian group). We show that these objects are in a natural 1-1 correspondence with the (≥ 0)-graded left EX-modules and that, for an object $(Y \to X)$ over X, the EX-module so corresponding to the strong abelianization of $(Y \to X)$ is the module $EX \otimes_{EY} IY$ (where $IY \subset EY$ denotes the augmentation ideal). As a result the Quillen homology groups $H_i^Q(X;M)$ and cohomology groups $H^i_Q(X;N)$ can be described by

$$H_i^Q(X;M) = \operatorname{Tor}_i^{EF.X}(M, IF.X),$$
$$H^i_Q(X;N) = \operatorname{Ext}^i_{EF.X}(IF.X, N), \qquad i \geq 0,$$

and the short exact sequence $0 \to IF.X \to EF.X \to Z \to 0$ therefore readily yields natural isomorphisms

$$H_i^Q(X;M) \approx H_{i+1}(X;M), \quad H^i_Q(X;N) \approx H^{i+1}(X;N), \qquad i \geq 1,$$

and natural exact sequences

$$0 \to H_1(X;M) \to H_0^Q(X;M) \to M \to H_0(X;M) \to 0,$$
$$0 \to H^0(X;N) \to N \to H^0_Q(X;N) \to H^1(X;N) \to 0.$$

1.3. **Notation, terminology, etc.** We will freely use notation, terminology and results of [3] and [7, Chapter II]. In particular:

(i) *Rings and modules.* All rings will have an identity and will be associative (but not necessarily commutative) and augmented over Z (the integers). Moreover, they will be (≥ 0)-graded, except that, for a simplicial ring R, its homotopy ring $\pi_* R$ will be bi-(≥ 0)-graded. All modules over bigraded rings will be bi-graded, and all other modules will be graded. This also applies to Z-modules, i.e., abelian groups.

(ii) *Whitehead products.* For a pointed connected CW complex L and elements $a \in \pi_{p+1}L$ and $b \in \pi_{q-1}L$ $(p, q \geq 1)$, we denote by $[a, b]$ the

Whitehead product which differs by $(-1)^p$ from the usual one [10, Chapter X]. As a result

$$[a, b] + (-1)^{pq}[b \cdot a] = 0$$

and if $c \in \pi_{r+1}L$ $(r \geq 1)$, then the Jacobi identity becomes

$$(-1)^{pr}[a, [b, c]] + (-1)^{pq}[b, [c, a]] + (-1)^{qr}[c, [a, b]] = 0.$$

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2. Π-ALGEBRAS

We start with recalling from [3] some facts about

2.1. **Π-algebras.** A *Π-algebra* consists [3, 2.3] of a (≥ 1)-graded group $\{X_p\}_{p=1}^{\infty}$, with X_p abelian for $p > 1$, together with three kinds of operations:

(i) a *Whitehead product* homomorphism $[-, -]: X_p \otimes X_q \to X_{p+q-1}$ for all $p, q > 1$, and

(ii) a *composition* function $(- \circ \alpha): X_p \to X_r$, for every element $\alpha \in \pi_r S^p$ with $r > p > 1$ (which need not be a homomorphism, but which is right additive, i.e., $(x \circ \alpha_1) + (x \circ \alpha_2) = (x \circ (\alpha_1 + \alpha_2))$ for all $x \in X_p$ and $\alpha_1, \alpha_2 \in \pi_r S^p$),

which satisfy all the relations that hold for the Whitehead product (1.3) and composition operations on the higher homotopy groups of pointed connected CW complexes, and

(iii) a *left action* of X_1 on the X_p $(p > 1)$ which commutes with the Whitehead product and composition operations (we will write $\tau_x y$ for the result of this left action by an element $x \in X_1$ on an element $y \in X_p$ $(p > 1)$).

Thus a Π-algebra X is completely determined by its universal covering $\widetilde{X}$ (i.e., the sub-Π-algebra $\widetilde{X} \subset X$ consisting of the elements of degree > 1) and the left action of the group X_1 on this Π-algebra $\widetilde{X}$.

The category of Π-algebras will be denoted by **Π-al**.

2.2. **Examples.** (i) The *homotopy* Π-*algebra* of a pointed connected CW complex. Let L be a pointed connected CW complex. Then the graded group $\{\pi_p L\}_{p=1}^{\infty}$, together with the usual Whitehead product (1.3) and composition operations and fundamental group action, is clearly a Π-algebra, which will be denoted by $\pi_* L$.

(ii) *Aspherical* Π-*algebras*. These are Π-algebras X such that (2.1) $\widetilde{X}$ is trivial. Clearly such an aspherical Π-algebra X is completely determined by the group X_1. Aspherical Π-algebras thus are just *groups*.

(iii) *Simply connected rational* Π-*algebras*. These are Π-algebras X such that $X_1 = 1$ and each X_p $(p > 1)$ is a rational vector space. Such Π-algebras are [3, 2.5] essentially *connected graded rational Lie algebras*; they are completely determined by their "underlying" connected graded rational Lie algebra, i.e., the Lie algebra obtained by lowering the degrees by 1 and taking the Whitehead product as Lie product.

(iv) *Strongly abelian* Π-*algebras*. These are Π-algebras X in which all Whitehead product and composition operations are trivial and in which X_1 is abelian and acts trivially on the X_p $(p > 1)$. Strongly abelian Π-algebras thus are just (≥ 1)-*graded abelian groups*.

2.3. **Free Π-algebras.** Another important class of Π-algebras is that of the *free* **Π**-*algebras*, i.e., **Π**-algebras which are isomorphic to homotopy **Π**-algebras of wedges of spheres. If $M = \bigvee_{j\in J} S^{p_j}$ $(p_j \geq 1)$, then π_*M is the free **Π**-algebra on the obvious generators $i_{p_j} \in \pi_{p_j}S^{p_j} \subset \pi_{p_j}M$ $(j \in J)$. Moreover the universal covering of π_*M is the homotopy **Π**-algebra of the universal covering of M, i.e., the free **Π**-algebra on the elements $\tau_x i_{p_j}$ with $x \in \pi_1 M$, $j \in J$, and $p_j > 1$, and $\pi_1 M$ acts on this **Π**-algebra from the left by permuting these generators in the obvious manner.

Clearly a sub-**Π**-algebra of a free **Π**-algebra need not be free. However, *if* U *and* V *are both free* **Π**-*algebras, then the kernel of the projection* $U \amalg V \to V$ *is also free.* To prove this one first notes: If $U_1 = V_1 = 1$ and $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ are sets of free generators for U and V respectively, then [3, 2.5(v); 6, Theorem 3] the kernel of the projection map $U \amalg V \to V$ is freely generated by the elements $[\cdots[a_i, b_{j_1}], \ldots, b_{j_n}]$ with $i \in I$, $n \geq 0$, and $j_1, \ldots, j_n \in J$.

2.4. **Simplicial Π-algebras.** The category **sΠ-al** of simplicial **Π**-algebras admits [7, II, §4] a closed simplicial model category structure in which the simplicial structure is the obvious one and in which a map is a fibration or a weak equivalence whenever the underlying map of simplicial sets is so. The cofibrant objects are the retracts of the free ones, where a simplicial **Π**-algebra Y is called *free* if there exists a subset $B \subset Y$, closed under the degeneracy operators, such that, for each $i \geq 0$, the **Π**-algebra of Y in dimension i is freely generated by the elements of B in dimension i. Moreover all simplicial **Π**-algebras are fibrant and hence [7, II, §2] *every weak equivalence* $f: Y \to Y' \in$ **sΠ-al** *between cofibrant objects is actually a homotopy equivalence*; i.e., there exists a map $f': Y' \to Y \in$ **sΠ-al** such that the compositions $f'f$ and ff' are simplicially homotopic to the identity maps of Y and Y' respectively.

2.5. **The standard resolution of a Π-algebra.** The *standard resolution* of a **Π**-algebra X consists of the free **Π**-algebra $F.X$ and weak equivalence $F.X \to X \in$ **sΠ-al**, in which each $(F.X)_n$ $(n \geq 0)$ consists of the **Π**-algebra $F^{n+1}X$ obtained by $(n+1)$-fold application of the free **Π**-algebra functor F (i.e., the forgetful functor from **Π-al** to ((≥ 1)-**graded pointed sets**) followed by its left adjoint) and in which the face and degeneracy operators and the map $F.X \to X$ are the obvious ones. Clearly this construction is functorial.

3. The enveloping ring of a Π-algebra

Next we briefly discuss the notion of

3.1. **The enveloping of a Π-algebra.** The *enveloping ring* of a **Π**-algebra X is [3, 4.1] the ring (1.3) EX which has, for every integer $p \geq 1$ and element $x \in X_p$, a generator ex in degree $p-1$, and which has the following relations:

$$\begin{aligned}
e(x+y) &= ex + ey, && x, y \in \widetilde{X}, \\
e[x, y] &= (ex)(ey) - (-1)^{pq}(ey)(ex), && x \in \widetilde{X}_{p+1},\ y \in \widetilde{X}_{q+1}, \\
e(xy) &= (ex)(ey), && x, y \in X_1, \\
e(\tau_x y) &= (ex)(ey)(ex^{-1}), && x \in X_1,\ y \in \widetilde{X}, \\
e(x \circ \alpha) &= H(\alpha)(ex)^2, && \alpha \in \pi_{2p-1}S^p,\ p \text{ even}, \\
&= 0, && \text{otherwise},
\end{aligned}$$

where $H(\alpha)$ denotes the Hopf invariant [3, 1.3(iv)] of α.

This definition immediately implies

3.2. Proposition. *Let $X \in$ **Π-al**. Then EX is the "semitensor product" of $E\widetilde{X}$ and ZX_1 (the integral group ring of X_1), in the sense that*

(i) *additively EX is actually isomorphic to $E\widetilde{X} \otimes ZX_1$,*
(ii) *the multiplication in EX is given by*

$$(ey \otimes ex)(ey' \otimes ex') = (ey)(e\tau_x y') \otimes (ex)(ex')$$

for all $x, x' \in X_1$ and $y, y' \in \widetilde{X}$.

3.3. Classical examples. (i) If $X \in$ **Π-al** is *aspherical* (2.2), then $EX = ZX_1$, the *integral group* ring of X_1, and the function $X_1 \to EX = ZX_1$ given by $x \to ex$ for all $x \in X_1$, is the usual map which sends the elements of X_1 to the corresponding (additive) generators of ZX_1.

(ii) If $X \in$ Π-al is *simply connected rational* (2.2), then EX is the *enveloping algebra of the underlying Lie algebra of* X (2.2), and the function $X \to EX$ given by $x \to ex$ for all $x \in X$, is the usual inclusion of one in the other.

Another easy consequence of Definition 3.1 is

3.4. Proposition. *Let X be a simply connected (i.e., $X_1 = 1$) free **Π**-algebra* (2.3) *and let $\{b_j\}_{j\in J}$ be a set of free generators for X. Then EX is the free (tensor) algebra on the elements eb_j $(j \in J)$. In particular, EX is additively just the free abelian group on the elements $(eb_{j_1})\cdots(eb_{j_n})$ with $n \geq 0$ and $j_1, \dots, j_n \in J$.*

3.5. Corollary (2.3 and 3.2). *Let X be a free **Π**-algebra and let $\{b_j\}_{j\in J}$ be a set of free generators for X. Then EX is additively the free abelian group on the elements $(e\tau_{x_1} b_{j_1})\cdots(e\tau_{x_n} b_{j_n})(ex)$ with $n \geq 0$, $x, x_1, \dots, x_n \in X_1$, $j_1, \dots, j_n \in J$, and $b_{j_1}, \dots, b_{j_n} \in \widetilde{X}$.*

A straightforward calculation now yields

3.6. Proposition. *Let X and $\{b_j\}_{j\in J}$ be as in 3.5. Then the augmentation ideal $IX \subset EX$ is a free left EX-module on the elements eb_j with $b_j \in \widetilde{X}$ and the elements $1 - eb_j$ with $b_j \in X_1$.*

We will also need

3.7. Proposition. *Let U and V be free **Π**-algebras with generators $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ respectively and let X be the kernel of the projection $U \amalg V \to V$. Then $E(U \amalg V)$ is a free right EX-module on the elements*

$$(e\tau_{v_1} b_{j_1})\cdots(e\tau_{v_n} b_{j_n})(ev)$$

with $n \geq 0$, $v, v_1, \dots, v_n \in V_1$, $j_1, \dots, j_n \in J$, and $b_{j_1}, \dots, b_{j_n} \in \widetilde{V}$.

Proof. If U and V are simply connected (i.e., $U_1 = V_1 = 1$), then one proves this by a lengthy but essentially straightforward calculation using 2.3. The general case now follows readily using 3.2 and 3.5.

We end with considering

3.8. **E-flat Π-algebras.** A Π-algebra X is [3, 5.2] called *E-flat* if (2.5) $\pi_i EF.X = 0$ for $i > 0$. Some examples of E-flat **Π**-algebras are, in view of [3, 5.4(i) and (ii)] and a slight variation on [3, 5.4(iii)]

(i) *aspherical* (2.2) **Π**-algebras,
(ii) *simply connected rational* (2.2) **Π**-algebras, and
(iii) *free* (2.3) **Π**-algebras.

4. The functors Tor_i^R for a simplicial ring R

In preparation for the definition (in §5) of the homology of a **Π**-algebra, we recall from [7, II, §6] some facts on simplicial modules over a simplicial ring R. In a slight change in notation we will write Tor_i^R for the functors which were denoted there by $\pi_i(- \otimes_R^L -)$.

4.1. **Simplicial modules over a simplicial ring.** Given a simplicial ring R (1.3), a *left simplicial R-module* consists of a simplicial abelian group B, together with a map of simplicial abelian groups $R \otimes B \to B$ which turns each B_i $(i \geq 0)$ into a left R-module. The resulting category of left simplicial R-modules will be denoted by $\mathbf{M}_R$.

Of course, *right simplicial R-modules* are just left simplicial R^{op}-modules (where R^{op} denotes the simplicial ring which in each dimension $i \geq 0$ consists of the opposite of the ring R_i) and the category of right simplicial R-modules is therefore denoted by $\mathbf{M}_{R^{\mathrm{op}}}$.

4.2. **A model category structure for $\mathbf{M}_R$.** The category $\mathbf{M}_R$ admits [7, II, §6] a closed simplicial model category structure in which the simplicial structure is the obvious one and in which a map is a fibration or a weak equivalence whenever the underlying map of simplicial sets is so. The cofibrant objects are the retracts of the free ones, where an object $B \in \mathbf{M}_R$ is called *free* if there exists a subset $X \subset B$, closed under the degeneracy operators, such that each B_i $(i \geq 0)$ is the free left R_i-module on $X \cap B_i$. Moreover, all objects of $\mathbf{M}_R$ are fibrant and it follows [7, II, §2] that every weak equivalence $f\colon B \to B' \in \mathbf{M}_R$ *between cofibrant objects is actually a homotopy equivalence*, i.e., there exists a map $f'\colon B' \to B \in \mathbf{M}_R$ such that the compositions $f'f$ and ff' are simplicially homotopic to the identity maps of B and B' respectively.

4.3. **Cofibrant resolutions.** Given $B \in \mathbf{M}_R$, a *cofibrant resolution* of B (over R) consists of a cofibrant object $B_\# \in \mathbf{M}_R$, together with a trivial fibration $j\colon B_\# \to B \in \mathbf{M}_R$ (i.e., a weak equivalence which is onto). Clearly (4.2) such cofibrant resolutions always exist. A convenient and functorial one is the *standard resolution* $R_\# B \to B \in \mathbf{M}_R$ in which, for each $i \geq 0$, $(R_\# B)_i = (R^{i+1}B)_i$, where $R^{i+1}B$ is obtained from B by $(i+1)$-fold application of the free left simplicial R-module functor R (i.e., the forgetful functor from $\mathbf{M}_R$ to (**pointed simplicial sets**), followed by its left adjoint) and in which the face and degeneracy maps and the map $R_\# B \to B \in \mathbf{M}_R$ are the obvious ones.

4.4. **The functor $\otimes_R$ for a simplicial ring R.** Given (4.1) simplicial modules $A \in \mathbf{M}_{R^{\mathrm{op}}}$ and $B \in \mathbf{M}_R$, the simplicial abelian group $A \otimes_R B$ (given by $(A \otimes_R B)_i = A_i \otimes_{R_i} B_i$ for all $i \geq 0$) *has homotopy meaning whenever B is cofibrant*, as a diagonal argument using [4] readily yields

4.5. **Proposition.** *Let* $A \in \mathbf{M}_{R^{\mathrm{op}}}$ *and let* $f: B \to B' \in \mathbf{M}_R$ *be a weak equivalence between cofibrant objects. Then the map* $A \otimes_R B \to A \otimes_R B'$ *induces isomorphisms* $\pi_i(A \otimes_R B) \approx \pi_i(A \otimes_R B')$ $(i \geq 0)$.

Remark. It is also useful to observe that if $f: A \to A' \in \mathbf{M}_{R^{\mathrm{op}}}$ is a weak equivalence and $B \in \mathbf{M}_R$ is a cofibrant object, then the map $f \otimes_R B: A \otimes_R B \to A' \otimes_R B$ induces isomorphisms $\pi_i(A \otimes_R B) \to \pi_i(A' \otimes_R B)$ $(i \geq 0)$. This follows from the analogue of 4.5 if A and A' are in the evident sense cofibrant objects of $\mathbf{M}_{R^{\mathrm{op}}}$, from the Corollary on page 6.10 of [7, II] if $f: A \to A'$ is a cofibrant resolution of A', and then in general by the technique of comparing f to the induced map of standard (functorial) cofibrant resolutions.

4.6. **The functors** Tor_i^R **for a simplicial ring** R. These functors will be somewhat different from the ones denoted by Tor_i^R in [7, II, §6]. Let $A \in \mathbf{M}_{R^{\mathrm{op}}}$ and $B \in \mathbf{M}_R$ and let $j: B_\# \to B$ be a cofibrant resolution (4.3) of B. Then *the abelian groups* $\pi_i(A \otimes_R B_\#)$ *do not depend on the choice of this cofibrant resolution and will be denoted by* $\mathrm{Tor}_i^R(A, B)$ (instead of by $\pi_i(A \otimes_R^L B)$ as in [7, II, §6]).

Similarly, for maps $a: A \to A' \in \mathbf{M}_{R^{\mathrm{op}}}$ and $b: B \to B' \in \mathbf{M}_R$, cofibrant resolutions $j: B_\# \to B$ and $j': B'_\# \to B'$ and a map $b_\#: B_\# \to B'_\# \in \mathbf{M}_R$ such that $j'b_\# = bj$ (which always exists), *the resulting maps*

$$\pi_i(a \otimes_R b_\#): \pi_i(A \otimes_R B_\#) = \mathrm{Tor}_i^R(A, B) \to \pi_i(A' \otimes_R B'_\#) = \mathrm{Tor}_i^R(A', B')$$

$(i > 0)$ *do not depend on the choices of* j, j' *and* $b_\#$ *and will be denoted by* $\mathrm{Tor}_i^R(a, b)$.

The functor Tor_i^R so defined are clearly functors from $\mathbf{M}_{R^{\mathrm{op}}} \times \mathbf{M}$ to (abelian groups) and one readily verifies

4.7. **Proposition.** *Let* $* \to A' \to A \to A'' \to *$ *and* $* \to B' \to B \to B'' \to *$ *be short exact sequences in* $\mathbf{M}_{R^{\mathrm{op}}}$ *and* $\mathbf{M}_R$ *respectively. Then there are natural long exact sequences*

$$\begin{aligned}\cdots \to \mathrm{Tor}_i^R(A', B) \to \mathrm{Tor}_i^R(A, B) \to \mathrm{Tor}_i^R(A'', B)&\\ \to \mathrm{Tor}_{i-1}^R(A', B) \to \cdots \to \mathrm{Tor}_0^R(A'', B)&,\\ \cdots \to \mathrm{Tor}_i^R(A, B') \to \mathrm{Tor}_i^R(A, B) \to \mathrm{Tor}_i^R(A, B'')&\\ \to \mathrm{Tor}_{i-1}^R(A, B') \to \cdots \to \mathrm{Tor}_0^R(A, B'')&.\end{aligned}$$

To get a further hold on the groups $\mathrm{Tor}_i^R(A, B)$, one notes [7, II, §6] that the multiplication on R turns the (≥ 0)-graded abelian group $\pi_* R$ into a ring, that the R-module structures on A and B turn the (≥ 0)-graded abelian groups $\pi_* A$ and $\pi_* B$ into right and left $\pi_* R$-modules respectively and that the resulting groups $\mathrm{Tor}_i^{\pi_* R}(\pi_* A, \pi_* B)$ are related to the groups $\mathrm{Tor}_i^R(A, B)$ by [7, II, §6].

4.8. **A Künneth spectral sequence.** *There is a natural first quadrant spectral sequence with*

$$E_{p,q}^2 = (\mathrm{Tor}_p^{\pi_* R}(\pi_* A, \pi_* B))_q \Rightarrow \mathrm{Tor}_{p+q}^R(A, B).$$

One also has [7, II, §6]

4.9. **A partial Künneth spectral sequence.** *There is a natural first quadrant spectral sequence with*

$$E^2_{p,q} = \operatorname{Tor}^R_p(\pi_q A, B) \Rightarrow \operatorname{Tor}^R_{p+q}(A, B)$$

where R *acts on each* $\pi_q A$ $(q \geq 0)$ *through the projection* $R \to \pi_0 R$.

5. Homology of Π-algebras

In this section we

(i) define, for a **Π**-algebra X and a right module M over the enveloping ring EX of X, *homology groups* $H_i(X; M)$ of X with coefficients in M, in such a manner that, for aspherical **Π**-algebras (i.e., groups) and for simply connected rational **Π**-algebras (which are essentially connected rational Lie algebras), this definition of homology coincides with the usual one, and

(ii) note that a short exact sequence of coefficient modules, as usual, gives rise to a *long exact sequence* of homology groups, while a short exact sequence $* \to U \to W \to V \to *$ of **Π**-algebras yields a *Serre spectral sequence* relating the homology groups of U and V to those of W.

We thus start with defining the

5.1. **Homology of Π-algebras.** Let X be a **Π**-algebra (2.1), let EX be its enveloping ring (3.1) and let M be a right EX-module. Then the *homology groups* $H_i(X; M)$ *of* X *with coefficients in* M are defined by (2.5 and 4.6)

$$H_i(X; M) = \operatorname{Tor}^{EF.X}_i(M, Z), \qquad i \geq 0,$$

where the simplicial ring $EF.X$ acts on M through the canonical map $EF.X \to EX$.

In view of 4.8 this definition implies

5.2. **Proposition.** *There is a natural first quadrant spectral sequence with*

$$E^2_{p,q} = (\operatorname{Tor}^{\pi_* EF.X}_p(M, Z))_q \Rightarrow H_{p+q}(X; M).$$

5.3. **Corollary.** *If* X *is* E*-flat* (3.8), *then there are natural isomorphisms*

$$\operatorname{Tor}^{EX}_i(M, Z) \approx H_i(X; M), \qquad i \geq 0.$$

Thus (3.8) for aspherical and for simply connected rational **Π**-algebras, the above definition of homology reduces to the usual one [5] for groups and for connected graded rational Lie algebras respectively.

Another immediate consequence of Definition 5.1 is (4.7):

5.4. **Proposition.** *Let* $* \to M' \to M \to M'' \to *$ *be a short exact sequence of right* EX*-modules. Then there is a natural long exact sequence*

$$\cdots \to H_i(X; M') \to H_i(X; M) \to H_i(X; M'') \to H_{i-1}(X; M') \to \cdots \to H_0(X; M'') \to 0.$$

Less trivial is the existence of

5.5. **A Serre spectral sequence.** *Let* $* \to U \to W \to V \to *$ *be a short exact sequence of* **Π***-algebras. Then there are, for every right* EW*-module* M,

(i) *a natural right* EV*-action on each* $H_i(U; M)$ $(i \geq 0)$, *and*

(ii) *a natural first quadrant spectral sequence with*

$$E^2_{p,q} = H_p(V\,;\, H_q(U\,;\, M)) \Rightarrow H_{p+q}(W\,;\, M)\,.$$

Proof. Let $F_{\cdot}'U$ be the kernel of the induced map $F_{\cdot}W \to F_{\cdot}V \in$ **sΠ-al** of standard resolutions (2.5). The arguments of 2.3 then readily imply that $F_{\cdot}'U$ is a free simplicial **Π**-algebra. It follows (2.4) that the inclusion $F_{\cdot}U \to F_{\cdot}'U \in$ **sΠ-al** is a homotopy equivalence and therefore [4] induces an isomorphism $\pi_* EF_{\cdot}U \approx \pi_* EF_{\cdot}'U$. The desired result now follows readily from 3.4, 3.7, 4.8, and

5.6. **Lemma.** *Let* $R \to S \to T$ *be a sequence of simplicial rings* (1.3) *such that*

(a) T *is a free simplicial* Z*-module,*
(b) S *is free as a right simplicial* R*-module, and*
(c) *the composition* $R \to T$ *factors through the augmentation* $R \to Z$ *and the induced map* $S \otimes_R Z \to T$ *is an isomorphism of left simplicial* S*-modules.*

Then there are, for every right simplicial S*-module* M,

(i) *a natural right action of* $\pi_0 T$ *(and hence* T*) on each* $\operatorname{Tor}_i^R(M, Z)$ $(i \geq 0)$,
(ii) *a natural first quadrant spectral sequence with*

$$E^2_{p,q} = \operatorname{Tor}_p^T(\operatorname{Tor}_q^R(M, Z), Z) \Rightarrow \operatorname{Tor}_{p+q}^S(M, Z)\,.$$

Proof. In view of (a) the canonical map (4.3) $(S \otimes_Z T^{\mathrm{op}})_\# T \to T$ is a cofibrant resolution of T over S and hence (4.8) the induced map $(S \otimes_Z T^{\mathrm{op}})_\# T \otimes_T T_\# Z \to Z$ is a cofibrant resolution of Z over S, so that, for all $i \geq 0$, $\pi_i(M \otimes_S (S \otimes_Z T^{\mathrm{op}})_\# T \otimes_T T_\# Z) \approx \operatorname{Tor}_i^S(M, Z)$. Furthermore (b), (c), and §4.3 imply that the composition

$$S \otimes_R R_\# Z = R_\# Z \otimes_{R^{\mathrm{op}}} S \to Z \otimes_{R^{\mathrm{op}}} S = S \otimes_R Z \approx T$$

is also a cofibrant resolution of T over S. It follows (4.2) that there are canonical isomorphisms

$$\begin{aligned}\operatorname{Tor}_i^R(M, Z) = \pi_i(M \otimes_R R_\# Z) &\approx \pi_i(M \otimes_S S \otimes_R R_\# Z) \\ &\approx \pi_i(M \otimes_S (S \otimes_Z T^{\mathrm{op}})_\# T)\end{aligned}$$

$(i \geq 0)$, which yield a natural right $\pi_0 T$-action on each $\operatorname{Tor}_i^R(M, Z)$ and the lemma now becomes an immediate consequence of §4.9.

6. The functors Ext_R^i for a simplicial ring R

In preparation for the definition (in §7) of the cohomology of a **Π**-algebra, we now "dualize" the results of §4. We start with defining

6.1. **Cosimplicial modules over a simplicial ring.** Given a simplicial ring R (1.3), a *left cosimplicial* R*-module* will consist of a cosimplicial abelian group C [1, Chapter X], together with a map of cosimplicial abelian groups $C \to \hom(R, C)$ (where $\hom(R, C)$ denotes the cosimplicial abelian group with $\hom(R, C)^i = \hom(R_i, C^i)$ for all $i \geq 0$ and with the obvious cosimplicial operators) which turns each C^i $(i \geq 0)$ into a left R_i-module. This is equivalent to requiring that each C^i $(i \geq 0)$ comes with a left R-module structure such that, for every pair of integers (h, i) with $0 \le h \le i$, the following diagrams are commutative:

$$
\begin{array}{ccccc}
 & & R_{i-1}\otimes C^{i-1} & \to & C^{i-1} \\
 & \nearrow^{d_h} & & & \\
R_i \otimes C^{i-1} & & & & \downarrow d^h \\
 & \searrow_{d^h} & & & \\
 & & R_i \otimes C^i & \to & C^i
\end{array}
$$

$$
\begin{array}{ccccc}
 & & R_{i+1}\otimes C^{i+1} & \to & C^{i+1} \\
 & \nearrow^{s_h} & & & \\
R_i \otimes C^{i+1} & & & & \downarrow s^h \\
 & \searrow_{s^h} & & & \\
 & & R_i \otimes C^i & \to & C^i
\end{array}
$$

The category of these left cosimplicial R-module will be denoted by $\mathbf{M}^R$.

6.2. **The functor** $\hom_R$ **for a simplicial ring** R. Given (4.1 and 6.1) modules $B \in \mathbf{M}_R$ and $C \in \mathbf{M}^R$, *the cosimplicial abelian group* $\hom_R(B, C)$ (with $\hom_R(B, C)^i = \hom_{R_i}(B_i, C^i)$ for all $i \ge 0$ and with the obvious cosimplicial operators) *has homotopy meaning when* B *is cofibrant* (4.2), as a diagonal argument using [4] readily yields

6.3. **Proposition.** *Let* $C \in \mathbf{M}^R$ *and let* $f: B \to B' \to \mathbf{M}_R$ *be a weak equivalence between cofibrant objects* (4.2). *Then the map* $\hom_R(f, C): \hom_R(B', C) \to \hom_R(B, C)$ *induces an isomorphism of cohomotopy groups* [1, Chapter X] $\pi^i\hom_R(B', C) \approx \pi^i\hom_R(B, C)$ $(i \ge 0)$.

Remark. It is useful to observe (see the remark after 4.5) that if $B \in \mathbf{M}_R$ is a cofibrant object and $f: C \to C' \in \mathbf{M}^R$ is a weak equivalence (i.e., a map inducing isomorphisms $\pi^* C \cong \pi^* C'$, $i \ge 0$), then the map

$$\hom_R(B, f): \hom_R(B, C) \to \hom_R(B, C')$$

gives isomorphisms $\pi^i\hom_R(B, C) \simeq \pi^i\hom_R(B, C')$ $(i \ge 0)$.

6.4. **The functors** Ext^i_R **for a simplicial ring** R. Let $C \in \mathbf{M}^R$ and $B \in \mathbf{M}_R$ and let $j: B_\# \to B$ be a cofibrant resolution of B (4.3). Then *the abelian groups* $\pi^i\hom_R(B_\#, C)$ $(i \ge 0)$ *do not depend on the choice of the cofibrant resolution and will be denoted by* $\mathrm{Ext}^i_R(B, C)$. Similarly, for maps $c: C' \to C \in \mathbf{M}^R$ and $b: B \to B' \in \mathbf{M}_R$, cofibrant resolutions $j: B_\# \to B$ and $j': B'_\# \to B'$ and a map $b_\#: B_\# \to B'_\# \in \mathbf{M}_R$ such that $jb_\# = bj'$ (which always exists) *the resulting maps*

$$
\begin{aligned}
\pi^i\hom_R(b_\#, c): \pi^i\hom_R(B'_\#, C') \\
= \mathrm{Ext}^i_R(B', C') \to \pi^i\hom_R(B_\#, C) = \mathrm{Ext}^i_R(B, C)
\end{aligned}
$$

$(i \ge 0)$ *do not depend on the choices of* j, j', *and* $b_\#$ *and will be denoted by* $\mathrm{Ext}^i_R(b, c)$.

Clearly the functions Ext^i_R so defined are functors from $\mathbf{M}^{\mathrm{op}}_R \times \mathbf{M}^R$ to (**abelian groups**) and one readily verifies

6.5. **Proposition.** *Let* $* \to B' \to B \to B'' \to *$ *and* $* \to C' \to C \to C'' \to *$ *be short exact sequences in* $\mathbf{M}_R$ *and* $\mathbf{M}^R$ *respectively. Then there are natural long exact sequences*

$$0 \to \operatorname{Ext}^0_R(B'', C) \to \cdots \to \operatorname{Ext}^i_R(B, C) \to \operatorname{Ext}^i_R(B, C) \to \operatorname{Ext}^i_R(B', C) \to \operatorname{Ext}^{i+1}_R(B'', C) \to,$$

$$0 \to \operatorname{Ext}^0_R(B, C') \to \cdots \to \operatorname{Ext}^i_R(B, C') \to \operatorname{Ext}^i_R(B, C) \to \operatorname{Ext}^i_R(B, C'') \to \operatorname{Ext}^{i+1}_r(B, C') \to .$$

To get a hold on the groups $\operatorname{Ext}^i_R(B, C)$ we need

6.6. **A natural action of $\pi_* R$ on $\pi^* C$.** Given a module $C \in \mathbf{M}^R$, a lengthy but essentially straightforward calculation yields that the maps $R_p \otimes C^n \to C^{n-p}$ $(0 \le p \le n)$ which send an element $x \otimes y \in R_p \otimes C^n$ to the element

$$\sum_{(\mu, \nu)} \varepsilon(\mu, \nu) s^\mu (s_\nu x) y \in C^{n-p}$$

(where (μ, ν) runs over all $(p, n-p)$ shuffles, i.e., permutations $(\mu_1, \ldots, \mu_p, \nu_1, \ldots, \nu_{n-p})$ of $(0, \ldots, n-1)$ such that $\mu_1 < \cdots < \mu_p$ and $\nu_1 < \cdots < \nu_{n-p}$, where $\varepsilon(\mu, \nu)$ denotes the sign of the permutation and where $s^\mu = s^{\mu_1} \cdots s^{\mu_p}$ and $s_\nu = s_{\nu_{n-p}} \cdots s_{\nu_1}$), gives rise to a *left action of* $\pi_* R$ *on the* (≤ 0)*-graded abelian group* $\pi^* C$ (i.e., we put (degree $\pi^i C) = -i$). The resulting groups $\operatorname{Ext}^i_{\pi_* R}(\pi_* B, \pi^* C)$ are then related to the groups $\operatorname{Ext}^i_R(B, C)$ by

6.7. **Künneth spectral sequence.** *There is a natural third quadrant spectral sequence with*

$$E_2^{p,q} = (\operatorname{Ext}^p_{\pi_* R}(\pi_* B, \pi^* C))_{-q} \Rightarrow \operatorname{Ext}^{p+q}_R(B, C).$$

One also has

6.8. **A partial Künneth spectral sequence.** *There is a natural third quadrant spectral sequence with*

$$E_2^{p,q} = \operatorname{Ext}^p_R(B, \pi^q C) \Rightarrow \operatorname{Ext}^{p+q}_R(B, C)$$

where R *acts on each* $\pi^q C$ $(q \ge 0)$ *through the projection* $R \to \pi_0 R$.

It remains to give

6.9. **Proofs of 6.7 and 6.8.** The proofs of 6.7 and 6.8 are similar to those of 4.8 and 4.9 [7, II, §6].

To prove 6.7, construct (as in [7, II, §6] or [9, §2]) a "simplicial left simplicial R-module" $V.B$ and a map $V.B \to B$ to the (discrete) simplicial left simplicial R-module B such that

(i) each left simplicial R-module $V_i B$ $(i \ge 0)$ is a free left simplicial R-module such that $\pi_* V_i B$ is a free left $\pi_* R$-module,

(ii) the induced map $\operatorname{diag} V.B \to B \in \mathbf{M}_R$ is a cofibrant resolution of B, and

(iii) the induced map $\pi_* V.B \to \pi_* B$ is a cofibrant (and in fact free) simplicial resolution of $\pi_* B$ over $\pi_* R$.

As $\hom_R(\operatorname{diag} V.B, C) \approx \operatorname{diag} \hom_R(V.B, C)$ and $\pi^* \hom_R(V_i B, C) \approx \hom_{\pi_* R}(\pi_* V_i B, \pi^* C)$ for all $i \ge 0$, the desired spectral sequence now is one of the spectral sequences of the bicosimplicial abelian group $\hom_R(V.B, C)$.

To prove 6.8 one constructs a Postnikov filtration on C, i.e., a filtration

$$0 = P_{-1}C \subset P_0C \subset \cdots \subset P_iC \subset \cdots \subset \bigcup_{n=1}^{\infty} P_nC = C$$

such that $\pi^i(P_iC/P_{i-1}C) = \pi^iC$ for all $i \geq 0$ and $\pi^j(P_iC/P_{i-1}C) = 0$ for $j \neq i$. This can be done, for instance, by constructing a corresponding filtration on the normalization NC of C [2, §3] and then denormalizing. The rest of the proof of 6.8 now is straightforward.

7. Cohomology of Π-algebras

"Dualizing" the results of §5 we

(i) define, for a **Π**-algebra X and a left module N over the enveloping ring EX of X, *cohomology groups* $H^i(X; N)$ of X with coefficients in N, in such a manner that for aspherical **Π**-algebras (i.e., groups) and simply connected rational **Π**-algebras (which are essentially connected graded rational Lie algebras) this definition of cohomology coincides with the usual one, and

(ii) note that a short exact sequence of coefficient modules, as usual, gives rise to a *long exact sequence* of cohomology groups, while a short exact sequence $* \to U \to W \to V \to *$ of **Π**-algebras yields a *Serre spectral sequence* relating the cohomology groups of U and V to those of W.

We thus start with defining the

7.1. **Cohomology of Π-algebras.** Let X be a **Π**-algebra (2.1), let EX be its enveloping ring (3.1) and let N be a left EX-module. Then the *cohomology groups* $H^i(X; N)$ of X with coefficients in N are defined by (2.5 and 6.4)

$$H^i(X; N) = \mathrm{Ext}^i_{EF.X}(Z, N)$$

where the simplicial ring $EF.X$ acts on N through the canonical map $EF.X \to EX$.

In view of 6.7 this definition implies

7.2. **Proposition.** *There is a natural third quadrant spectral sequence with*

$$E_2^{p,q} = (\mathrm{Ext}^p_{\pi_* EF.X}(Z, N))_{-q} \Rightarrow H^{p+q}(X; N).$$

7.3. **Corollary.** *If X is E-flat* (3.8), *then there are natural isomorphisms*

$$\mathrm{Ext}^i_{EX}(Z, N) \approx H^i(X; M), \qquad i \geq 0.$$

Thus (3.8) for aspherical and for simply connected rational **Π**-algebras, the above definition of cohomology reduces to the usual ones [5] for groups and for connected graded rational Lie algebras respectively.

Another immediate consequence of Definition 7.1 is (6.5)

7.4. **Proposition.** *Let $* \to N' \to N \to N'' \to *$ be a short exact sequence of left EX-modules. Then there is a natural long exact sequence*

$$0 \to H^0(X; N') \to \cdots \to H^i(X; N') \to H^i(X; N) \to H^i(X; N'') \to H^{i+1}(X; N') \to \cdots.$$

Less trivial is again the proof of the existence of

7.5. **A Serre spectral sequence.** Let $* \to U \to W \to V \to *$ be a short exact sequence of **Π**-algebras. Then there are, for every left EW-module N,

(i) a natural left EV-action on each $H^i(U; N)$ $(i \geq 0)$,
(ii) a natural third quadrant spectral sequence with

$$E_2^{p,q} = H^p(V; H^q(U; N)) \Rightarrow H^{p+q}(W; N).$$

Proof. This is essentially the same as the proof of 5.5, using (instead of 4.8 and 5.6) 6.7 and

7.6. **Lemma.** *Let* $R \to S \to T$ *be a sequence of simplicial rings such that* 5.6(a)–(c) *hold. Then there are, for every left cosimplicial* S*-module* N*,*

(i) *a natural left action of* $\pi_0 T$ *(and hence* T*) on each* $\operatorname{Ext}^i_R(Z, N)$ $(i \geq 0)$,
(ii) *a natural third quadrant spectral sequence with*

$$E_2^{p,q} = \operatorname{Ext}^p_T(Z, \operatorname{Ext}^q_R(Z, N)) \Rightarrow \operatorname{Ext}^{p+q}_S(Z, N).$$

Proof. By the arguments of 5.6 one has

$$\pi^{p+q}\hom_S((S \otimes_Z T^{\mathrm{op}})_\# T \otimes_T T_\# Z, N) \cong \operatorname{Ext}^{p+q}_S(Z, N)$$

and

$$\pi^q \hom_S((S \otimes_Z T^{\mathrm{op}})_\# T, N) \cong \operatorname{Ext}^q_S(T, N) \cong \pi^q \hom_S(S \otimes_S R_\# Z, N) \cong \operatorname{Ext}^q_R(Z, N).$$

The result follows from 6.8 with T in place of R, $T_\# Z$ in place of R, and $\hom_S((S \otimes_Z T^{\mathrm{op}})_\# T, N)$ in place of C.

8. Quillen homology and cohomology

We end with a Quillen-like approach to the homology and cohomology of **Π**-algebras and note that (except in the bottom dimension) the resulting groups differ from the ones of §5 and §7 by only a shift in dimension. To do all this we need the notion of

8.1. **Strong abelianization over a Π-algebra.** Given a **Π**-algebra X, let $(\mathbf{\Pi\text{-}al}/X)$ denote its over category (which has as objects the maps $Y \to X \in \mathbf{\Pi\text{-}al}$ and as maps the obvious commutative triangles) and let $(\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}}$ denote the category of the *abelian group objects* in $(\mathbf{\Pi\text{-}al}/X)$ which have a *strongly abelian* (2.2) *kernel.* In other words, the objects of $(\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}}$ are the diagrams

$$* \to B \xrightarrow{i} Y' \underset{j}{\overset{p}{\rightleftarrows}} X \to *$$

in **Π-al** in which

(i) B is strongly abelian,
(ii) the right-pointing arrows form an exact sequence,
(iii) $pj = \mathrm{id}\colon X \to X$.

The *strong abelianization functor*

$$\mathrm{sab}\colon (\mathbf{\Pi\text{-}al}/X) \to (\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}}$$

then is defined as the left adjoint of the forgetful functor $(\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}} \to (\mathbf{\Pi\text{-}al}/X)$.

To "compute", for an object $(Y \to X) \in (\mathbf{\Pi\text{-}al}/X)$, the image $\mathrm{sab}(Y \to X) \in (\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}}$, we need

8.2. **Proposition.** (i) *Given an object of* $(\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}}$, *i.e., a diagram*

$$* \to B \xrightarrow{i} Y' \underset{j}{\overset{p}{\rightleftarrows}} X \to *$$

in $\mathbf{\Pi\text{-}al}$ *as in* 8.1, *let* $B^{\#}$ *denote the* (≥ 0)*-graded abelian group obtained from* B *by lowering the degrees by* 1, *and, for every element* $b \in B$, *let the same symbol* b *denote the corresponding element of* $B^{\#}$. *Then there is a unique left* EX*-module structure on* $B^{\#}$ *such that, if we identify the elements* $x \in X$ *and* $b \in B$ *with their images in* Y' *under* i *and* j *respectively,*

$$\begin{aligned} (ex)b &= xbx^{-1}, & & x \in X_1,\ b \in \mathbf{B}_1, \\ &= \tau_x b, & & x \in \mathbf{X}_1,\ b \in \widetilde{B}, \\ &= x - \tau_b x, & & x \in \widetilde{\mathbf{X}},\ b \in \mathbf{B}_1, \\ &= [x, b], & & x \in \widetilde{X},\ b \in \widetilde{B}. \end{aligned}$$

(ii) *Moreover, the resulting functor*

$$\mathbf{\Phi}\colon (\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}} \to ((\geq 0)\text{-}\mathbf{graded\ left}\ EX\text{-}\mathbf{modules})$$

is an equivalence of categories.

Now we can state

8.3. **Proposition.** *Let* $(Y \to X) \in (\mathbf{\Pi\text{-}al}/x)$. *Then*

$$\mathbf{\Phi}(\mathrm{sab}(Y \to X)) = EX \otimes_{EY} IY,$$

where $IY \subset EY$ *denotes the augmentation ideal. In particular*

$$\mathbf{\Phi}(\mathrm{sab}(\mathrm{id}\colon X \to X)) = IX.$$

As the proofs of these two propositions are rather long, we will postpone them until §9, and proceed here with defining

8.4. **The Quillen homology of a Π-algebra.** Given a Π-algebra X and a right EX-module M, one can consider the composite functor $H^Q(-; M)$

$$(\mathbf{\Pi\text{-}al}/X) \xrightarrow{\mathrm{sab}} (\mathbf{\Pi\text{-}al}/X)_{\mathrm{sab}} \xrightarrow{\mathbf{\Phi}} \mathbf{M}_{EX^{\mathrm{op}}} \xrightarrow{M \otimes_{EX} -} (\mathbf{abelian\ groups})$$

which (8.3) send an object $(Y \to X) \in (\mathbf{\Pi\text{-}al}/X)$ to the abelian group $M \otimes_{EX} EX \otimes_{EY} IY = M \otimes_{EY} IY$. The *Quillen homology groups of* X *with coefficients in* M then will be defined as the left derived functors of the functor $H^Q(-; M)$, applied to the identity map of X, i.e. (2.4, 2.5, and [8]),

$$H_i^Q(X; M) = \pi_i(M \otimes_{EF.X} IF.X)$$

or equivalently (3.6 and 4.6)

$$H_i^Q(X; M) = \mathrm{Tor}_i^{EF.X}(M, IF.X).$$

Moreover, combining this with 4.7, 5.1, and the short exact sequence $* \to IF.X \to EF.X \to Z \to *$ one gets

8.5. **Proposition.** *There are natural isomorphisms*

$$H_i^Q(X\,;M) \approx H_{i+1}(X\,;M), \qquad i>0,$$

and a natural exact sequence

$$0 \to H_1(X\,;M) \to H_0^Q(X\,;M) \to M \to H_0(X\,;M) \to 0.$$

"Dualizing" all this one obtains

8.6. **The Quillen cohomology of a Π-algebra.** Given a Π-algebra X and a left EX-module N, one can consider the composite functor $H_Q(-\,;N)$

$$(\mathbf{\Pi\text{-al}}/X) \xrightarrow{\text{sab}} (\mathbf{\Pi\text{-al}}/X)_{\text{sab}} \xrightarrow{\Phi} \mathbf{M}_{EX} \xrightarrow{\hom_{EX}(-\,,N)} (\textbf{abelian groups})$$

which (8.3) sends an object $(Y \to X) \in (\mathbf{\Pi\text{-al}}/X)$ to the abelian group

$$\hom_{EX}(EX \otimes_{EY} IY\,, N) = \hom_{EY}(IY\,, N).$$

The *Quillen cohomology groups of* X with coefficients in N then will be defined as the left derived functors of the functor $H_Q(-\,;N)$, applied to the identity map of X, i.e. (2.4, 2.5, and [8]),

$$H_Q^i(X\,;N) = \pi^i \hom_{EF.X}(IF.X\,, N)$$

or equivalently (3.6 and 6.4)

$$H_Q^i(X\,;N) = \operatorname{Ext}^i_{EF.X}(IF.X\,, N).$$

Moreover combining this with 6.5, 7.1, and the short exact sequence $* \to IF.X \to EF.X \to Z \to *$ one gets

8.7. **Proposition.** *There are natural isomorphisms*

$$H_Q^i(X\,;N) \approx H^{i+1}(X\,;N), \qquad i>0,$$

and a natural exact sequence

$$0 \to H^0(X\,;N) \to N \to H_Q^0(X\,;N) \to H^1(X\,;N) \to 0.$$

It thus remains to give

9. Proofs of Propositions 8.2 and 8.3

We start with a

9.1. **Proof of 8.2(i).** As EX is generated by the ex $(x \in X)$, there is at most one such left EX-module structure on $B^{\#}$ and one thus has to show only that the formulas of 8.2 indeed determine such a structure. Most of this is straightforward. The only nontrivial part is to show that, for $b \in B_q$, $x \in X_p$, and $\alpha \in \pi_r S^p$ $(r > p > 1)$

$$\begin{aligned} e(x \circ \alpha)b &= H(\alpha)(ex)^2 b, \quad && p \text{ even and } r = 2p-1, \\ &= 0, && \text{otherwise}. \end{aligned}$$

If $q > 1$, one notes [3, 1.3] that, unless p is even and $r = 2p-1$, the element (2.3) $[\alpha\,, i_q] \in \pi_{q+r-1}(S^p \vee S^q)$ has finite order and hence [3, 3.4] is a finite sum of $(i_p\,, i_q)$-decomposable (i.e., elements which are Whitehead products in i_p and i_q, followed by a composition). Naturality then yields that $e(x \circ \alpha)b = 0$.

If p is even and $r = 2p - 1$, then similarly $[\alpha, i_q] = H(\alpha)[i_p, [i_p, i_q]]$ modulo (i_p, i_q)-decomposable, which implies that $e(x \circ \alpha)b = H(\alpha)(ex)^2 b$.

If $q = 1$, one considers the element $(\alpha - \tau\alpha) \in \pi_r(S^p \vee S^1)$, where $\tau = \tau_{i_1}$. Again, unless p is even and $r = 2p - 1$, this element has finite order and hence is a finite sum of $(i_p, \tau i_p)$-decomposables, which implies that $e(x \circ \alpha)b = 0$. If p is even and $r = 2p - 1$, then [10, Chapter XI]

$$(\alpha - \tau\alpha) + H(\alpha)[i_p - \tau i_p, i_p - \tau i_p] = (i_p - \tau i_p) \circ \alpha + H(\alpha)[i_p, i_p - \tau i_p]$$

and from this one readily deduces that $e(x \circ \alpha)b = H(\alpha)(ex)^2 b$.

9.2. **Proof of 8.2(ii).** The main problem here is to construct a potential inverse for the functor $\mathbf{\Phi}$. To do this we recall from [3, §2] that a $\mathbf{\Pi}$-algebra can be considered as a contravariant functor from the category of finitely generated free $\mathbf{\Pi}$-algebras to the category of pointed sets, which sends coproducts to products. Given a (≥ 0)-graded left EX-module A, we then consider the contravariant functor which sends a finitely generated free $\mathbf{\Pi}$-algebra V to the (pointed) set of pairs (f, g), where f is a map $f: V \to X \in \mathbf{\Pi}\text{-}\mathbf{al}$ and g is a map of left EV-modules $g: IV \to (Ef)^* A$. In view of 3.6 this functor sends coproducts to products and hence determines a $\mathbf{\Pi}$-algebra, say Y', and one readily verifies that there are obvious maps $p: Y' \to X$ and $j: X \to Y' \in \mathbf{\Pi}\text{-}\mathbf{al}$ such that $pj =$ id and such that the kernel of p is the strongly abelian $\mathbf{\Pi}$-algebra, obtained from A by raising the degrees by 1. It remains to show that the resulting functor to $(\mathbf{\Pi}\text{-}\mathbf{al}/X)_{\text{sab}}$ is actually a two-sided inverse of $\mathbf{\Phi}$, but that is straightforward.

We end, as promised, with a

9.3. **Proof of 8.3.** Let

$$* \to B \xrightarrow{i} Y' \underset{j}{\overset{p}{\rightleftarrows}} X \to *$$

and $B^{\#}$ be as in 8.2. Given an object $f: Y \to X \in (\mathbf{\Pi}\text{-}\mathbf{al}/X)$ and a map $k: EX \otimes_{EY} IY \to B^{\#}$ of left EX-modules, let $tk: Y \to Y'$ be the function defined (in the notation of 8.2) by

$$\begin{aligned}(tk)y &= k(1 \otimes (ey - 1))fy, \qquad & y \in \mathbf{Y}_1, \\ &= k(1 \otimes ey) + fy, & y \in \widetilde{Y}.\end{aligned}$$

Then a straightforward calculation using [10, XI, 8.5] shows that tk is actually a map in $\mathbf{\Pi}\text{-}\mathbf{al}$ such that $p(tk) = f$. Moreover the function t so defined is, in view of 3.6, a natural 1-1 correspondence between the left EX-module maps $EX \otimes_{EY} IY \to B^{\#}$ and the maps $f \to p \in (\mathbf{\Pi}\text{-}\mathbf{al}/X)$, whenever Y is a free $\mathbf{\Pi}$-algebra, and a direct limit argument readily implies that this restriction on Y is superfluous. Thus the functor which send an object $f: Y \to X \in (\mathbf{\Pi}\text{-}\mathbf{al}/X)$ to the left EX-module $EX \otimes_{EY} IY$ is left adjoint to the composition of $\mathbf{\Phi}^{-1}$ and the forgetful functor $(\mathbf{\Pi}\text{-}\mathbf{al}/X)_{\text{sab}} \to (\mathbf{\Pi}\text{-}\mathbf{al}/X)$ and hence canonically naturally equivalent to the functor $\mathbf{\Phi}(\text{sab})$.

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